Math 245C Lecture 13 Notes

Daniel Raban

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1 Derivatives of Convolutions

1.1 L^p and weak L^p convolution inequalities

Last time, we proved the first part of the following theorem:

Theorem 1.1. *Let $1 \leq p, q, r \leq \infty$ be such that $1 + r^{-1} = p^{-1} + q^{-1}$. Let $f \in L^p$.*

1. *(Generalized Young's inequality) If $g \in L^q$, then*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

2. *Further assume $1 < p, q, r < \infty$ and $g \in \text{weak } L^q$, Then there is a constant $C_{p,q}$ independent of f, g such that*

$$\|f * g\|_r \leq C_{p,q} \|f\|_p [g]_q.$$

3. *If $p = 1$ (so $q = r < \infty$), there exists a constant C_q independent of f such that for any $g \in \text{weak } L^q$,*

$$[f * g]_r \leq C_p \|f\|_1 [g]_q.$$

Proof. To complete the proof of the theorem, observe that $[K(x, \cdot)]_q = [g(x - \cdot)]_q = [g]_q < \infty$. Similarly, $[K(\cdot, y)]_q = [g]_q < \infty$. By our interpolation theorem for kernel operators with $c = [g]_q$, we have

$$\|Tf\|_r \leq c\|f\|_p, \qquad p > 1,$$
$$[Tf]_r \leq CB\|f\|_1, \qquad p = 1, r = q.$$

□

1.2 Convolution of C^k functions

Proposition 1.1. *Let $f \in C^k$ be such that $\partial^\alpha f$ is bounded for any $|\alpha| \le k$, and let $g \in L^1$. Then $f * g \in C^k$ and $\partial^\alpha(f * g) = \partial^\alpha f * g$*

Proof. Proceed by induction on $|\alpha|$. Assume then that $|\alpha| = 1$. Note that

$$f(x+h) = f(x) + \nabla f(x) \cdot h + \int_0^1 (\nabla f(x+th) - \nabla f(x)) \cdot d\, dt.$$

Hence,

$$\begin{aligned} f * g(x+h) &= \int_{\mathbb{R}^n} f(x+h-y)g(y)\, dy \\ &= \int_{\mathbb{R}^n} f(x-y)g(y)\, dy + h \cdot \int_{\mathbb{R}^n} \nabla f(x-y)g(y)\, dy + h \cdot \int_{\mathbb{R}^n} \int_0^1 \varepsilon(t,y)g(y)\, dt\, dy, \end{aligned}$$

where $\varepsilon(t,y) = \nabla f(x-y+th) - \nabla f(x-y)$. Note that $\|\varepsilon\|_u \le 2\|\nabla f\|_u$. Thus,

$$|\varepsilon(t,y)g(y)| \le |g(y)|, \qquad g \in L^1((0,1) \times \mathbb{R}^n).$$

So

$$\lim_{h\to 0} \int_{\mathbb{R}^n} \int_0^1 \varepsilon(t,y)g(y)\, dt\, dy = \int_{\mathbb{R}^n} \int_0^1 \lim_{h\to 0} \varepsilon(t,y)g(y)\, dt\, dy = 0.$$

In other words,

$$f * g(x+h) = g * g(x) + h \cdot (\nabla f * g)(x) + h \cdot \gamma(x,h), \qquad \lim_{h\to 0} |\gamma(x,h)| = 0.$$

This proves that $\nabla(f * g)$s exists and equals $\nabla f * g$. □

1.3 Convolution of functions in the Schwarz space

Proposition 1.2. *If $f, g \in \mathcal{S}$, then $f * g \in \mathcal{S}$.*

Proof. By the previous proposition, $f, g \in C^\infty$. Recall that $\|f\|_{(N,\alpha)} = \|(1+|x|)^N \partial \alpha f\|_u$ and that these are bounded for all α, N. Note that

$$(1+|x|) \le 1 + |x-y| + |y| \le (1+|x-y|)(1+|y|),$$

and so

$$\begin{aligned} (1+|x|)^N |\partial^\alpha(f * g(x))| &= (1+|x|)^N |(\partial^\alpha f) * g(x)| \\ &\le \int_{\mathbb{R}^n} (1+|x-y|)^N |\partial^\alpha f(x-y)| (1+|y|)^N |g(y)|\, dy \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} (1+|x-y|)^N |\partial^\alpha f(x-y)| (1+|y|)^{N+n+1} |g(y)| \frac{1}{(1-|y|)^{n+1}}\, dy \\
&\le \int_{\mathbb{R}^n} \|f\|_{(N,\alpha)} \|g\|_{(N+n+1,0)} \frac{1}{(1+|y|)^{n+1}}\, dy \\
&= |S^{n-1}| \|f\|_{(N,\alpha)} \|g\|_{(N+n+1,0)} \int_0^\infty \frac{r^{n-1}}{(1+r)^{n+1}}\, dr \\
&< \infty. \qquad \square
\end{aligned}
$$

Remark 1.1. If μ be a measure and $\int_{\mathbb{R}^n} f(x-y)\, d\mu(y)$ makes sense, we denote it as $f * \mu$.

Example 1.1. Let $\mu = \delta_a$. Then

$$
\int_{\mathbb{R}^n} \varphi(y)\, d\mu(y) = \varphi(a),
$$

and so

$$
f * \delta_a(x) = f(x-a).
$$

Let $g \in L^1$, and set

$$
g_t(x) = \frac{1}{t^n} g(x/t).
$$

Then

$$
\int_{\mathbb{R}^n} g_t(x)\, dx = \int_{\mathbb{R}^n} g(x/t) d(x/t) = a = \int_{\mathbb{R}^n} g(y)\, dy.
$$

We get

$$
\int_{\mathbb{R}^n} \varphi(y)\, g_t(y)\ , dy = \int_{\mathbb{R}^n} \varphi(y) g(y/t)\, d(y/t) = \int_{\mathbb{R}^n} \varphi(tz) g(z)\, dz \xrightarrow{t \to 0} \varphi(0) \int_{\mathbb{R}^n} g(z)\, dz = \varphi(0) a.
$$

So $g_t \to a\delta_0$.